SHORT REVISION
PARABOLA

1. CONIC SECTIONS:
A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.
- The fixed point is called the **Focus.**
- The fixed straight line is called the **Directrix.**
- The constant ratio is called the **Eccentricity** denoted by e.
- The line passing through the focus & perpendicular to the directrix is called the **Axis**.
- A point of intersection of a conic with its axis is called a **Vertex**.

2. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY :
The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :
$(l^2 + m^2) [(x - p)^2 + (y - q)^2] = e^2 (lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3. DISTINGUISHING BETWEEN THE CONIC :
The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) : When The Focus Lies On The Directrix.
In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if :
$e > 1$ the lines will be real & distinct intersecting at S.
$e = 1$ the lines will coincident.
$e < 1$ the lines will be imaginary.

Case (II) : When The Focus Does Not Lie On Directrix.

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1$; $D \neq 0$,	$0 < e < 1$; $D \neq 0$;	$e > 1$; $D \neq 0$;	$e > 1$; $D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$; $a + b = 0$

4. PARABOLA : DEFINITION :
A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).
Standard equation of a parabola is $y^2 = 4ax$. For this parabola :
(i) Vertex is (0, 0) (ii) focus is (a, 0) (iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$

FOCAL DISTANCE :
The distance of a point on the parabola from the focus is called the **Focal Distance Of The Point.**

FOCAL CHORD :
A chord of the parabola, which passes through the focus is called a **Focal Chord.**

DOUBLE ORDINATE :
A chord of the parabola perpendicular to the axis of the symmetry is called a **Double Ordinate.**

LATUS RECTUM :
A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **Latus Rectum.** For $y^2 = 4ax$.
- Length of the latus rectum = 4a.
- ends of the latus rectum are L(a, 2a) & L' (a, – 2a).

Note that:
(i) Perpendicular distance from focus on directrix = half the latus rectum.
(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.
(iii) Two parabolas are laid to be equal if they have the same latus rectum.
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

5. POSITION OF A POINT RELATIVE TO A PARABOLA :
The point $(x_1\ y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

6. LINE & *A* PARABOLA :
The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtrless cm \Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

7.
Length of the chord intercepted by the parabola on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note: length of the focal chord making an angle α with the x– axis is $4a\text{Cosec}^2\alpha$.

8. PARAMETRIC REPRESENTATION :

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$. The equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\, y + 2\, at_1 t_2 = 0$.

Note: If the chord joining t_1, t_2 & t_3, t_4 pass through a point (c, 0) on the axis, then $t_1t_2 = t_3t_4 = -c/a$.

9. TANGENTS TO THE PARABOLA $y^2 = 4ax$:

(i) $y y_1 = 2a(x + x_1)$ at the point (x_1, y_1) ; **(ii)** $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(iii) $ty = x + at^2$ at $(at^2, 2at)$.

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[\, at_1 t_2,\, a(t_1 + t_2)\,]$.

10. NORMALS TO THE PARABOLA $y^2 = 4ax$:

(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; **(ii)** $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

Note : Point of intersection of normals at t_1 & t_2 are, $a(t_1^2 + t_2^2 + t_1t_2 + 2)\,;\, -a t_1 t_2 (t_1 + t_2)$.

11. THREE VERY IMPORTANT RESULTS :

(a) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(c) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1 t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

General Note :

(i) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.

(ii) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum.

(iii) If a family of straight lines can be represented by an equation $\lambda^2P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

12. The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y y_1 - 2a(x + x_1)$.

13. DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **Director Circle.** It's equation is $x + a = 0$ which is parabola's own directrix.

14. CHORD OF CONTACT :

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $(y_1^2 - 4ax_1)^{3/2} \div 2a$. Also note that the chord of contact exists only if the point P is not inside.

15. POLAR & POLE :

(i) Equation of the Polar of the point $P(x_1, y_1)$ w.r.t. the parabola $y^2 = 4ax$ is,

$y y_1 = 2a(x + x_1)$

(ii) The pole of the line $lx + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

Note:

(i) The polar of the focus of the parabola is the directrix.

(ii) When the point (x_1, y_1) lies without the parabola the equation to its polar is the same as the equation to the chord of contact of tangents drawn from (x_1, y_1) when (x_1, y_1) is on the parabola the polar is the same as the tangent at the point.

(iii) If the polar of a point P passes through the point Q, then the polar of Q goes through P.

(iv) Two straight lines are said to be conjugated to each other w.r.t. a parabola when the pole of one lies on the other.

(v) Polar of a given point P w.r.t. any Conic is the locus of the harmonic conjugate of P w.r.t. the two points is which any line through P cuts the conic.

16. CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$. This reduced to $T = S_1$ where $T \equiv y y_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

17. DIAMETER :

The locus of the middle points of a system of parallel chords of a Parabola is called a **Diameter**. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

Note:

(i) The tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.

(ii) The tangent at the ends of any chords of a parabola meet on the diameter which bisects the chord.

(iii) A line segment from a point P on the parabola and parallel to the system of parallel chords is called the ordinate to the diameter bisecting the system of parallel chords and the chords are called its double ordinate.

18. IMPORTANT HIGHLIGHTS :

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus.**

(c) The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangtent at the vertex.

(e) If the tangents at P and Q meet in T, then :

- TP and TQ subtend equal angles at the focus S.
- $ST^2 = SP. SQ$ &
- The triangles SPT and STQ are similar.

(f) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.

(g) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $2a = \frac{2bc}{b+c}$ **i.e.** $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(i) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co-ordinates $-a, a(t_1 + t_2 + t_3 + t_1t_2t_3)$.

(j) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(k) If normal drawn to a parabola passes through a point P(h, k) then $k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

Then gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1 m_2 m_3 = -\frac{k}{a}$.

where m_1, m_2 & m_3 are the slopes of the three concurrent normals. Note that the algebraic sum of the:

- slopes of the three concurrent normals is zero.
- ordinates of the three conormal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the x-axis.

(*l*) A circle circumscribing the triangle formed by three co–normal points passes through the vertex of the parabola and its equation is, $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$

***Suggested problems from Loney*: Exercise-25** (Q.5, 10, 13, 14, 18, 21), **Exercise-26 (Important)** (Q.4, 6, 7, 16, 17, 20, 22, 26, 27, 28, 34, 38), **Exercise-27** (Q.4, 7), **Exercise-28** (Q.2, 7, 11, 14, 17, 23), **Exercise-29** (Q.7, 8, 10, 19, 21, 24, 26, 27), **Exercise-30** (2, 3, 13, 18, 20, 21, 22, 25, 26, 30)

***Note*:** Refer to the figure on Pg.175 if necessary.

EXERCISE–1

Q.1 Show that the normals at the points (4a, 4a) & at the upper end of the latus ractum of the parabola $y^2 = 4ax$ intersect on the same parabola.

Q.2 Prove that the locus of the middle point of portion of a normal to $y^2 = 4ax$ intercepted between the curve & the axis is another parabola. Find the vertex & the latus rectum of the second parabola.

Q.3 Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact.

Q.4 A circle is described whose centre is the vertex and whose diameter is three-quarters of the latus rectum of a parabola $y^2 = 4ax$. Prove that the common chord of the circle and parabola bisects the distance between the vertex and the focus.

Q.5 Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2,5).

Q.6 Through the vertex O of a parabola $y^2 = 4x$, chords OP & OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ.

Q.7 Let S is the focus of the parabola $y^2 = 4ax$ and X the foot of the directrix, PP' is a double ordinate of the curve and PX meets the curve again in Q. Prove that P'Q passes through focus.

Q.8 Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that one of the normals is given by $y = x - 3$ & find the equations of the others.

Q.9 Find the equations of the chords of the parabola $y^2 = 4ax$ which pass through the point $(-6a, 0)$ and which subtends an angle of 45° at the vertex.

Q.10 Through the vertex O of the parabola $y^2 = 4ax$, a perpendicular is drawn to any tangent meeting it at P & the parabola at Q. Show that $OP \cdot OQ$ = constant.

Q.11 'O' is the vertex of the parabola $y^2 = 4ax$ & L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H, prove that the length of the double ordinate through H is $4a\sqrt{5}$.

Q.12 The normal at a point P to the parabola $y^2 = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that $QG^2 - PG^2$ = constant.

Q.13 If the normal at P(18, 12) to the parabola $y^2 = 8x$ cuts it again at Q, show that $9PQ = 80\sqrt{10}$

Q.14 Prove that, the normal to $y^2 = 12x$ at (3, 6) meets the parabola again in (27, –18) & circle on this normal chord as diameter is $x^2 + y^2 - 30x + 12y - 27 = 0$.

Q.15 Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point (6, 9).

Q.16 P & Q are the points of contact of the tangents drawn from the point T to the parabola $y^2 = 4ax$. If PQ be the normal to the parabola at P, prove that TP is bisected by the directrix.

Q.17 Prove that the locus of the middle points of the normal chords of the parabola $y^2 = 4ax$ is $\frac{y^2}{2a} + \frac{4a^3}{y^2} = x - 2a$.

Q.18 From the point (–1, 2) tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact & the tangents.

Q.19 Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola.

Q.20 From a point A common tangents are drawn to the circle $x^2 + y^2 = a^2/2$ & parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola.

Q.21 Prove that on the axis of any parabola $y^2 = 4ax$ there is a certain point K which has the property that, if a chord PQ of the parabola be drawn through it, then $\frac{1}{(PK)^2} + \frac{1}{(QK)^2}$ is same for all positions of the chord. Find also the coordinates of the point K.

Q.22 Prove that the two parabolas $y^2 = 4ax$ & $y^2 = 4c(x - b)$ cannot have a common normal, other than the axis, unless $\frac{b}{(a-c)} > 2$.

Q.23 Find the condition on 'a' & 'b' so that the two tangents drawn to the parabola $y^2 = 4ax$ from a point are normals to the parabola $x^2 = 4by$.

Q.24 Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y^2(2x + a) = a(3x + a)^2$.

Q.25 Show that the locus of a point, such that two of the three normals drawn from it to the parabola $y^2 = 4ax$ are perpendicular is $y^2 = a(x - 3a)$.

EXERCISE–2

Q.1 In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus ractum meets the axis in T & the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5.

Q.2 Two tangents to the parabola $y^2= 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2 = 8(x + 2)$.

Q.3 A variable chord $t_1 t_2$ of the parabola $y^2 = 4ax$ subtends a right angle at a fixed point t_0 of the curve. Show that it passes through a fixed point. Also find the co–ordinates of the fixed point.

Q.4 Two perpendicular straight lines through the focus of the parabola $y^2 = 4ax$ meet its directrix in T & T' respectively. Show that the tangents to the parabola parallel to the perpendicular lines intersect in the mid point of T T '.

Q.5 Two straight lines one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$ are right angles. Find the locus of their point of intersection.

Q.6 A variable chord PQ of the parabola $y^2 = 4x$ is drawn parallel to the line y = x. If the parameters of the points P & Q on the parabola are p & q respectively, show that p + q = 2. Also show that the locus of the point of intersection of the normals at P & Q is 2x – y = 12.

Q.7 Show that an infinite number of triangles can be inscribed in either of the parabolas $y^2 = 4ax$ & $x^2 = 4by$ whose sides touch the other.

Q.8 If (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points on the parabola $y^2 = 4ax$ and the normals at these points meet in a point then prove that $\frac{x_1 - x_2}{y_3} + \frac{x_2 - x_3}{y_1} + \frac{x_3 - x_1}{y_2} = 0$.

Q.9 Show that the normals at two suitable distinct real points on the parabola $y^2 = 4ax$ intersect at a point on the parabola whose abscissa > 8a.

Q.10 The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a.
(a) Prove that each of these parabolas pass through a common point and determine the coordinates of this point.
(b) The vertices of the parabolas lie on a curve. Prove that this curve is a parabola and find its equation.

Q.11 The normals at P and Q on the parabola $y^2 = 4ax$ intersect at the point R (x_1, y_1) on the parabola and the tangents at P and Q intersect at the point T. Show that,
$$l(TP) \cdot l(TQ) = \frac{1}{2}(x_1 - 8a)\sqrt{y_1^2 + 4a^2}$$
Also show that, if R moves on the parabola, the mid point of PQ lie on the parabola $y^2 = 2a(x + 2a)$.

Q.12 If Q (x_1, y_1) is an arbitrary point in the plane of a parabola $y^2 = 4ax$, show that there are three points on the parabola at which OQ subtends a right angle, where O is the origin. Show furhter that the normal at these three points are concurrent at a point R.,determine the coordinates of R in terms of those of Q.

Q.13 PC is the normal at P to the parabola $y^2 = 4ax$, C being on the axis. CP is produced outwards to Q so that PQ = CP; show that the locus of Q is a parabola, & that the locus of the intersection of the tangents at P & Q to the parabola on which they lie is $y^2 (x + 4a) + 16 a^3 = 0$.

Q.14 Show that the locus of the middle points of a variable chord of the parabola $y^2 = 4ax$ such that the focal distances of its extremities are in the ratio 2 : 1, is $9(y^2 – 2ax)^2 = 4a^2(2x – a)(4x + a)$.

Q.15 A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola.

Q.16 Prove that the parabola $y^2 = 16x$ & the circle $x^2 + y^2 – 40x – 16y – 48 = 0$ meet at the point P(36, 24) & one other point Q. Prove that PQ is a diameter of the circle. Find Q.

Q.17 A variable tangent to the parabola $y^2 = 4ax$ meets the circle $x^2 + y^2 = r^2$ at P & Q. Prove that the locus of the mid point of PQ is $x(x^2 + y^2) + ay^2 = 0$.

Q.18 Find the locus of the foot of the perpendicular from the origin to chord of the parabola $y^2 = 4ax$ subtending an angle of 45^0 at the vertex.

Q.19 Show that the locus of the centroids of equilateral triangles inscribed in the parabola $y^2 = 4ax$ is the parabola $9y^2 – 4ax + 32 a^2 = 0$.

Q.20 The normals at P, Q, R on the parabola $y^2 = 4ax$ meet in a point on the line y = k. Prove that the sides of the triangle PQR touch the parabola $x^2 – 2ky = 0$.

Q.21 A fixed parabola $y^2 = 4ax$ touches a variable parabola. Find the equation to the locus of the vertex of the variable parabola. Assume that the two parabolas are equal and the axis of the variable parabola remains parallel to the x-axis.

Q.22 Show that the circle through three points the normals at which to the parabola $y^2 = 4ax$ are concurrent at the point (h, k) is $2(x^2 + y^2) - 2(h + 2a)x - ky = 0$.

Q.23 Prove that the locus of the centre of the circle, which passes through the vertex of the parabola $y^2 = 4ax$ & through its intersection with a normal chord is $2y^2 = ax - a^2$.

Q.24 The sides of a triangle touch $y^2 = 4ax$ and two of its angular points lie on $y^2 = 4b(x + c)$. Show that the locus of the third angular point is $a^2y^2 = 4(2b - a)^2.(ax + 4bc)$

Q.25 Three normals are drawn to the parabola $y^2 = 4ax \cos\alpha$ from any point lying on the straight line $y = b \sin\alpha$. Prove that the locus of the orthocentre of the traingles formed by the corresponding tangents is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the angle α being variable.

EXERCISE–3

Q.1 Find the locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other. [REE '97, 6]

Q.2 The angle between a pair of tangents drawn from a point P to the parabola $y^2 = 4ax$ is 45^0. Show that the locus of the point P is a hyperbola. [JEE '98, 8]

Q.3 The ordinates of points P and Q on the parabola $y^2 = 12x$ are in the ratio 1 : 2. Find the locus of the point of intersection of the normals to the parabola at P and Q. [REE '98, 6]

Q.4 Find the equations of the common tangents of the circle $x^2 + y^2 - 6y + 4 = 0$ and the parabola $y^2 = x$. [REE '99, 6]

Q.5(a) If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of ' k ' is
(A) 1/8 (B) 8 (C) 4 (D) 1/4

(b) If $x + y = k$ is normal to $y^2 = 12x$, then ' k ' is : [JEE'2000 (Scr), 1+1]
(A) 3 (B) 9 (C) – 9 (D) – 3

Q.6 Find the locus of the points of intersection of tangents drawn at the ends of all normal chords of the parabola $y^2 = 8(x - 1)$. [REE '2001, 3]

Q.7(a) The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x – axis is
(A) $\sqrt{3}\,y = 3x + 1$ (B) $\sqrt{3}\,y = -(x + 3)$ (C) $\sqrt{3}\,y = x + 3$ (D) $\sqrt{3}\,y = -(3x + 1)$

(b) The equation of the directrix of the parabola, $y^2 + 4y + 4x + 2 = 0$ is
(A) $x = -1$ (B) $x = 1$ (C) $x = -\frac{3}{2}$ (D) $x = \frac{3}{2}$
[JEE'2001(Scr), 1+1]

Q.8 The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix [JEE'2002 (Scr.), 3]
(A) $x = -a$ (B) $x = -\frac{a}{2}$ (C) $x = 0$ (D) $x = \frac{a}{2}$

Q.9 The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is [JEE'2002 (Scr), 3]
(A) $3y = 9x + 2$ (B) $y = 2x + 1$
(C) $2y = x + 8$ (D) $y = x + 2$

Q.10(a) The slope of the focal chords of the parabola $y^2 = 16x$ which are tangents to the circle $(x - 6)^2 + y^2 = 2$ are
(A) ± 2 (B) – 1/2, 2 (C) ± 1 (D) – 2, 1/2
[JEE'2003, (Scr.)]

(b) Normals are drawn from the point 'P' with slopes m_1, m_2, m_3 to the parabola $y^2 = 4x$. If locus of P with $m_1 m_2 = \alpha$ is a part of the parabola itself then find α. [JEE 2003, 4 out of 60]

Q.11 The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is
(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$
[JEE 2004, (Scr.)]

Q.12 Let P be a point on the parabola $y^2 - 2y - 4x + 5 = 0$, such that the tangent on the parabola at P intersects the directrix at point Q. Let R be the point that divides the line segment QP externally in the ratio $\frac{1}{2} : 1$. Find the locus of R. [JEE 2004, 4 out of 60]

Q.13(a) The axis of parabola is along the line y = x and the distance of vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the 1st quadrant, then the equation of the parabola is

(A) $(x + y)^2 = (x - y - 2)$ (B) $(x - y)^2 = (x + y - 2)$
(C) $(x - y)^2 = 4(x + y - 2)$ (D) $(x - y)^2 = 8(x + y - 2)$

[JEE 2006, 3]

(b) The equations of common tangents to the parabola $y = x^2$ and $y = -(x - 2)^2$ is/are

(A) $y = 4(x - 1)$ (B) $y = 0$ (C) $y = -4(x - 1)$ (D) $y = -30x - 50$

[JEE 2006, 5]

(c) **Match the following**

Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then

(i) Area of ΔPQR (A) 2
(ii) Radius of circumcircle of ΔPQR (B) 5/2
(iii) Centroid of ΔPQR (C) (5/2, 0)
(iv) Circumcentre of ΔPQR (D) (2/3, 0)

[JEE 2006, 6]

KEY CONCEPTS

ELLIPSE

1. **STANDARD EQUATION & DEFINITIONS :**

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

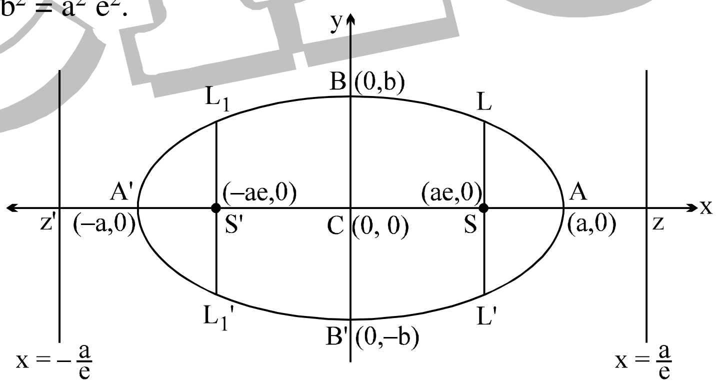
Where a > b & $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2 e^2$.

Where e = eccentricity (0 < e < 1).

FOCI : $S \equiv (a\,e, 0)$ & $S' \equiv (-a\,e, 0)$.

EQUATIONS OF DIRECTRICES :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.



VERTICES :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

MAJOR AXIS :

The line segment A′A in which the foci S′ & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z).**

MINOR AXIS :

The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment B′B of length 2b (b < a) is called the **Minor Axis** of the ellipse.

PRINCIPAL AXIS :

The major & minor axis together are called **Principal Axis** of the ellipse.

CENTRE :

The point which bisects every chord of the conic drawn through it is called the **centre** of the conic.

$C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

DIAMETER :

A chord of the conic which passes through the centre is called a **diameter** of the conic.

FOCAL CHORD : A chord which passes through a focus is called a **focal chord**.

DOUBLE ORDINATE :

A chord perpendicular to the major axis is called a **double ordinate.**

LATUS RECTUM :

The focal chord perpendicular to the major axis is called the **latus rectum**. Length of latus rectum

(LL′) $= \frac{2b^2}{a} = \frac{(minor\ axis)^2}{major\ axis} = 2a(1-e^2) = 2e$ (distance from focus to the corresponding directrix)

NOTE :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e. BS = CA.**

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned, then the rule is to assume that $a > b$.

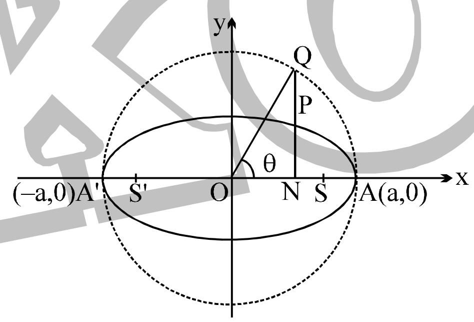
2. POSITION OF A POINT w.r.t. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >< \text{or} = 0$.

3. AUXILIARY CIRCLE / ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the **CORRESPONDING POINTS** on the ellipse & the auxiliary circle respectively 'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.



Note that $\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{Semi\ minor\ axis}{Semi\ major\ axis}$

Hence " If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

4. PARAMETRIC REPRESENTATION :

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Where θ is a parameter. Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ; $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

5. LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is $<$ = or $>$ $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$.

6. TANGENTS :

(i) $\frac{x x_1}{a^2}+\frac{y y_1}{b^2}=1$ is tangent to the ellipse at (x_1, y_1).

Note : The figure formed by the tangents at the extremities of latus rectum is rhoubus of area $\frac{2a^2}{e}$

(ii) $y = mx \pm\sqrt{a^2m^2+b^2}$ is tangent to the ellipse for all values of m.

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(iii) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

(iv) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is p then the tangents at these points are parallel.

(v) Point of intersection of the tangents at the point α & β is $a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$.

7. NORMALS :

(i) Equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ is ; $ax\sec\theta - by\,\mathrm{cosec}\,\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $y = mx-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

8. DIRECTOR CIRCLE : Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle**. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

9. Chord of contact, pair of tangents, chord with a given middle point, pole & polar are to be interpreted as they are in parabola.

10. DIAMETER :

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation $y = -\frac{b^2}{a^2m}x$.

11. IMPORTANT HIGHLIGHTS : Refering to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

H – 1 If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.

H – 2 The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars Y,Y′ lie on its auxiliary circle.The tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similiar ellipse as that of the original one. Also the lines joining centre to the feet of the perpendicular Y and focus to the point of contact of tangent are parallel.

H – 3 If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively,

& if CF be perpendicular upon this normal then

(i) PF. PG = b^2 **(ii)** PF. Pg = a^2 **(iii)** PG. Pg = SP. S′P **(iv)** CG. CT = CS^2

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse. [where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

H – 4 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.

H – 5 The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

H – 6 The circle on any focal distance as diameter touches the auxiliary circle.

H – 7 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

H – 8 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then,
(i) T t. PY = $a^2 - b^2$ and (ii) least value of Tt is $a + b$.

***Suggested problems from Loney*: Exercise-32** (Q.2 to 7, 11, 12, 16, 24), **Exercise-33 (Important)** (Q.3, 5, 6, 15, 16, 18, 19, 24, 25, 26, 34), **Exercise-35** (Q.2, 4, 6, 7, 8, 11, 12, 15)

EXERCISE–4

Q.1 Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and passing through the point (4, 6).

Q.2 The tangent at any point P of a circle $x^2 + y^2 = a^2$ meets the tangent at a fixed point A (a, 0) in T and T is joined to B, the other end of the diameter through A, prove that the locus of the intersection of AP and BT is an ellipse whose ettentricity is $\frac{1}{\sqrt{2}}$.

Q.3 The tangent at the point α on a standard ellipse meets the auxiliary circle in two points which subtends a right angle at the centre. Show that the eccentricity of the ellipse is $(1 + \sin^2\alpha)^{-1/2}$.

Q.4 An ellipse passes through the points (– 3, 1) & (2, –2) & its principal axis are along the coordinate axes in order. Find its equation.

Q.5 If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}\cdot\tan\frac{\delta}{2}=1$, where α, β, γ, δ are the eccentric angles of the extremities of the chords.

Q.6 If the normals at the points P, Q, R with eccentric angles α, β, γ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then show that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix} = 0$$

Q.7 Prove that the equation to the circle, having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the ends of a latus rectum, is $x^2 + y^2 - 2ae^3x = a^2(1 - e^2 - e^4)$.

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Q.8 Find the equations of the lines with equal intercepts on the axes & which touch the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Q.9 The tangent at $P\left(4\cos\theta, \frac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x - 15 = 0$. Find θ. Find also the equation to the common tangent.

Q.10 A tangent having slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$, intersects the axis of x & y in points A & B respectively. If O is the origin, find the area of triangle OAB.

Q.11 'O' is the origin & also the centre of two concentric circles having radii of the inner & the outer circle as 'a' & 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the two circles. If the focii of this ellipse lie on the inner circle, find the ratio of inner : outer radii & find also the eccentricity of the ellipse.

Q.12 ABC is an isosceles triangle with its base BC twice its altitude. A point P moves within the triangle such that the square of its distance from BC is half the rectangle contained by its distances from the two sides. Show that the locus of P is an ellipse with eccentricity $\sqrt{\frac{2}{3}}$ passing through B & C.

Q.13 Let d be the perpendicular distance from the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to the tangent drawn at a point P on the ellipse. If F_1 & F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left[1-\frac{b^2}{d^2}\right]$.

Q.14 Common tangents are drawn to the parabola $y^2 = 4x$ & the ellipse $3x^2 + 8y^2 = 48$ touching the parabola at A & B and the ellipse at C & D. Find the area of the quadrilateral.

Q.15 If the normal at a point P on the ellipse of semi axes a, b & centre C cuts the major & minor axes at G & g, show that $a^2. (CG)^2 + b^2. (Cg)^2 = (a^2 - b^2)^2$. Also prove that $CG = e^2CN$, where PN is the ordinate of P.

Q.16 Prove that the length of the focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is inclined to the major axis at angle θ is $\frac{2ab^2}{a^2\sin^2\theta+b^2\cos^2\theta}$.

Q.17 The tangent at a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersects the major axis in T & N is the foot of the perpendicular from P to the same axis. Show that the circle on NT as diameter intersects the auxiliary circle orthogonally.

Q.18 The tangents from (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intersect at right angles. Show that the normals at the points of contact meet on the line $\frac{y}{y_1}=\frac{x}{x_1}$.

Q.19 Find the locus of the point the chord of contact of the tangent drawn from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

touches the circle $x^2 + y^2 = c^2$, where $c < b < a$.

Q.20 Prove that the three ellipse $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 1$ and $\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} = 1$ will have a common tangent if $\begin{vmatrix} a_1^2 & b_1^2 & 1 \\ a_2^2 & b_2^2 & 1 \\ a_3^2 & b_3^2 & 1 \end{vmatrix} = 0$.

EXERCISE–5

Q.1 PG is the normal to a standard ellipse at P, G being on the major axis. GP is produced outwards to Q so that PQ = GP. Show that the locus of Q is an ellipse whose eccentricity is $\frac{a^2 - b^2}{a^2 + b^2}$ & find the equation of the locus of the intersection of the tangents at P & Q.

Q.2 P & Q are the corresponding points on a standard ellipse & its auxiliary circle. The tangent at P to the ellipse meets the major axis in T. Prove that QT touches the auxiliary circle.

Q.3 The point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is joined to the ends A, A′ of the major axis. If the lines through P perpendicular to PA, PA′ meet the major axis in Q and R then prove that
l(QR) = length of latus rectum.

Q.4 Let S and S' are the foci, SL the semilatus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and LS' produced cuts the ellipse at P, show that the length of the ordinate of the ordinate of P is $\frac{(1 - e^2)}{1 + 3e^2} a$, where 2a is the length of the major axis and e is the eccentricity of the ellipse.

Q.5 A tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches at the point P on it in the first quadrant & meets the coordinate axis in A & B respectively. If P divides AB in the ratio 3 : 1 find the equation of the tangent.

Q.6 PCP′ is a diameter of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b) & QCQ′ is the corresponding diameter of the auxiliary circle, show that the area of the parallelogram formed by the tangent at P, P', Q & Q' is $\frac{8a^2 b}{(a - b)\sin 2\phi}$ where ϕ is the eccentric angle of the point P.

Q.7 If the normal at the point P(θ) to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$, intersects it again at the point Q(2θ), show that $\cos\theta = -(2/3)$.

Q.8 A normal chord to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes an angle of 45° with the axis. Prove that the square of its length is equal to $\frac{32a^4 b^4}{(a^2 + b^2)^3}$

Q.9 If (x_1, y_1) & (x_2, y_2) are two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the tangents at which meet in (h, k) & the normals in (p, q), prove that $a^2 p = e^2 h x_1 x_2$ and $b^4 q = -e^2 k y_1 y_2 a^2$ where 'e' is the eccentricity.

Q.10 A normal inclined at 45° to the axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is drawn. It meets the x-axis & the y-axis in P & Q respectively. If C is the centre of the ellipse, show that the area of triangle CPQ is $\frac{(a^2 - b^2)^2}{2(a^2 + b^2)}$ sq. units.

Q.11 Tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from the point $\left(\frac{a^2}{\sqrt{a^2-b^2}}, \sqrt{a^2+b^2}\right)$. Prove that they intercept on the ordinate through the nearer focus a distance equal to the major axis.

Q.12 P and Q are the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If the chord P and Q touches the ellipse $\frac{4x^2}{a^2}+\frac{y^2}{b^2}-\frac{4x}{a}=0$, prove that $\sec\alpha+\sec\beta=2$ where α, β are the eccentric angles of the points P and Q.

Q.13 A straight line AB touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the circle $x^2 + y^2 = r^2$; where $a > r > b$. A focal chord of the ellipse, parallel to AB intersects the circle in P & Q, find the length of the perpendicular drawn from the centre of the ellipse to PQ. Hence show that PQ = 2b.

Q.14 Show that the area of a sector of the standard ellipse in the first quadrant between the major axis and a line drawn through the focus is equal to 1/2 ab $(\theta - e \sin\theta)$ sq. units, where θ is the eccentric angle of the point to which the line is drawn through the focus & e is the eccentricity of the ellipse.

Q.15 A ray emanating from the point (– 4, 0) is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection.

Q.16 If p is the length of the perpendicular from the focus 'S' of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on any tangent at 'P', then show that $\frac{b^2}{p^2}=\frac{2a}{\ell(SP)}-1$.

Q.17 In an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, n_1 and n_2 are the lengths of two perpendicular normals terminated at the major axis then prove that : $\frac{1}{n_1^2}+\frac{1}{n_2^2}=\frac{a^2+b^2}{b^4}$

Q.18 If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity 'e' of the ellipse is given by the absolute value of $\frac{\cos\beta}{\cos\alpha}$.

Q.19 Using the fact that the product of the perpendiculars from either foci of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ upon a tangent is b^2, deduce the following loci. An ellipse with 'a' & 'b' as the lengths of its semi axes slides between two given straight lines at right angles to one another. Show that the locus of its centre is a circle & the locus of its foci is the curve, $(x^2 + y^2)(x^2 y^2 + b^4) = 4 a^2 x^2 y^2$.

Q.20 If tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercept on the x-axis a constant length c, prove that the locus of the point of intersection of tangents is the curve
$4y^2 (b^2x^2 + a^2y^2 - a^2b^2) = c^2 (y^2 - b^2)^2$.

EXERCISE–6

Q.1 If tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $(\sqrt{5}\cos\phi, 2\sin\phi)$ on the ellipse $4x^2 + 5y^2 = 20$. Find the values of t & ϕ. [REE '96, 6]

Q.2 A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P & Q. Prove that the tangents at P & Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. [JEE '97, 5]

Q.3(i) The number of values of c such that the straight line $y = 4x + c$ touches the curve $(x^2/4) + y^2 = 1$ is
(A) 0 (B) 1 (C) 2 (D) infinite

(ii) If P = (x, y), F_1 = (3, 0), F_2 = (–3, 0) and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals
(A) 8 (B) 6 (C) 10 (D) 12

[JEE '98, 2 + 2]

Q.4(a) If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) & (x_3, y_3) :
(A) lie on a straight line (B) lie on on ellipse (C) lie on a circle (D) are vertices of a triangle.

(b) On the ellipse, $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are :
(A) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (B) $\left(-\frac{2}{5}, \frac{1}{5}\right)$ (C) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (D) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

(c) Consider the family of circles, $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of the family and the ellipse $4x^2 + 25y^2 = 100$ meets the co–ordinate axes at A & B, then find the equation of the locus of the mid–point of AB. [JEE '99, 2 + 3 + 10 (out of 200)]

Q.5 Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$. [REE '99, 6]

Q.6 Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ meet the ellipse respectively at P, Q, R so that P, Q, R lie on the same side of the major axis as A, B, C respectively. Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. [JEE '2000, 7]

Q.7 Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C. [JEE '2001, 5]

Q.8 Find the condition so that the line $px + qy = r$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\frac{\pi}{4}$. [REE '2001, 3]

Q.9 Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact must on the corresponding directrix. [JEE ' 2002, 5]

Q.10(a) The area of the quadrilateral formed by the tangents at the ends of the latus rectum of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is
(A) $9\sqrt{3}$ sq. units (B) $27\sqrt{3}$ sq. units (C) 27 sq. units (D) none

(b) The value of θ for which the sum of intercept on the axis by the tangent at the point $\left(3\sqrt{3}\cos\theta, \sin\theta\right)$, $0 < \theta < \pi/2$ on the ellipse $\frac{x^2}{27} + y^2 = 1$ is least, is : [JEE ' 2003 (Screening)]
(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{8}$

Q.11 The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2 + 2y^2 = 2$, between the coordinates axes, is
(A) $\frac{1}{x^2} + \frac{1}{2y^2} = 1$ (B) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$ (C) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (D) $\frac{1}{2x^2} + \frac{1}{y^2} = 1$
[JEE 2004 (Screening)]

Q.12(a) The minimum area of triangle formed by the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and coordinate axes is
(A) ab sq. units (B) $\frac{a^2 + b^2}{2}$ sq. units (C) $\frac{(a+b)^2}{2}$ sq. units (D) $\frac{a^2 + ab + b^2}{3}$ sq. units
[JEE 2005 (Screening)]

(b) Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Also find the length of the intercept of the tangent between the coordinate axes.

[JEE 2005 (Mains), 4]

KEY CONCEPTS
HYPERBOLA

The **HYPERBOLA** is a conic whose eccentricity is greater than unity. ($e > 1$).

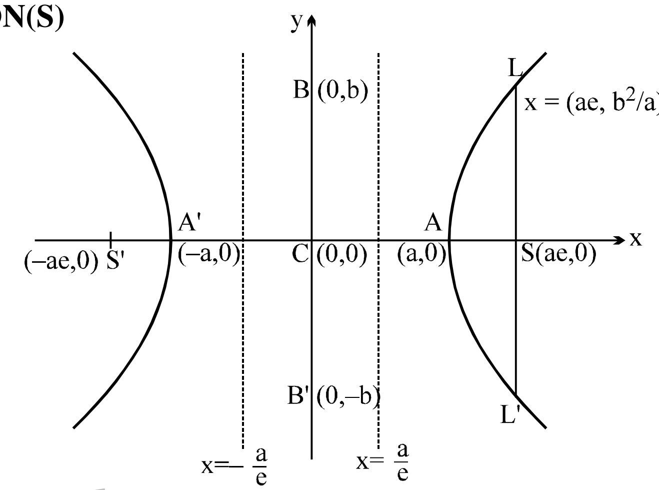
1. STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Where $b^2 = a^2 (e^2 - 1)$

or $a^2 e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \left(\frac{C.A}{T.A}\right)^2$



FOCI :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

EQUATIONS OF DIRECTRICES :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES : $A \equiv (a, 0)$ & $A' \equiv (-a, 0)$.

***l* (Latus rectum)** $= \frac{2b^2}{a} = \frac{(C.A.)^2}{T.A.} = 2a(e^2 - 1)$.

Note : l (L.R.) = 2e (distance from focus to the corresponding directrix)

TRANSVERSE AXIS : The line segment A′A of length 2a in which the foci S′ & S both lie is called the **T.A. OF THE HYPERBOLA**.

CONJUGATE AXIS : The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the **C.A. OF THE HYPERBOLA**.

The T.A. & the C.A. of the hyperbola are together called the Principal axes of the hyperbola.

2. FOCAL PROPERTY :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left||PS| - |PS'|\right| = 2a$. The distance SS' = focal length.

3. CONJUGATE HYPERBOLA :

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called **CONJUGATE HYPERBOLAS** of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each.

Note That :
(a) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.
(c) Two hyperbolas are said to be similiar if they have the same eccentricity.

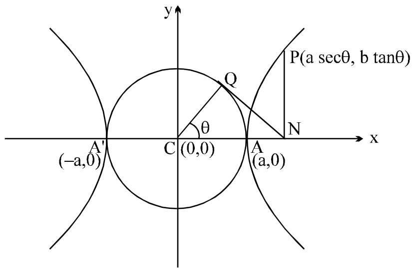
4. RECTANGULAR OR EQUILATERAL HYPERBOLA :

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an **EQUILATERAL HYPERBOLA**. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of its latus rectum is equal to its transverse or conjugate axis.

5. AUXILIARY CIRCLE :

A circle drawn with centre C & T.A. as a diameter is called the **AUXILIARY CIRCLE** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the **"CORRESPONDING POINTS "** on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. ($0 \le \theta < 2\pi$).



Note : The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations : $x = a \cos h\, \phi$, $y = b \sin h\, \phi$ also represents the same hyperbola.

General Note : Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2.

6. POSITION OF A POINT 'P' w.r.t. A HYPERBOLA :

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or without the curve.

7. LINE AND A HYPERBOLA :

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according as: $c^2 > = < a^2 m^2 - b^2$.

8. TANGENTS AND NORMALS :

TANGENTS :

(a) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$.

Note: In general two tangents can be drawn from an external point $(x_1\, y_1)$ to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1\, y_1)$ to the hyperbola.

(b) Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$.

Note : Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\frac{\theta_1 - \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}}$, $y = b\frac{\sin\frac{\theta_1 + \theta_2}{2}}{\cos\frac{\theta_1 + \theta_2}{2}}$

(c) $y = mx \pm \sqrt{a^2m^2 - b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note that there are two parallel tangents having the same slope m.

(d) Equation of a chord joining α & β is

$$\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$$

NORMALS:

(a) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(x_1, y_1)$ on it is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 - b^2 = a^2 e^2$.

(b) The equation of the normal at the point P $(a \sec\theta, b \tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a x}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2 e^2$.

(c) Equation to the chord of contact, polar, chord with a given middle point, pair of tangents from an external point is to be interpreted as in ellipse.

9. DIRECTOR CIRCLE :

The locus of the intersection of tangents which are at right angles is known as the **DIRECTOR CIRCLE** of the hyperbola. The equation to the director circle is :

$$x^2 + y^2 = a^2 - b^2.$$

If $b^2 < a^2$ this circle is real; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve.

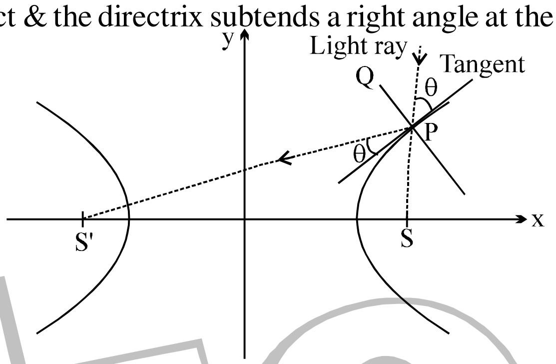
If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

10. HIGHLIGHTS ON TANGENT AND NORMAL :

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 \cdot (\text{semi } C \cdot A)^2$

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H–3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray"** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.



Note that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{a^2 - k^2} - \frac{y^2}{k^2 - b^2} = 1$ $(a > k > b > 0)$ Xare confocal and therefore orthogonal.

H–4 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

11. ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as

$$(b^2 - a^2m^2)\, x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0 \qquad ....(1)$$

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are :

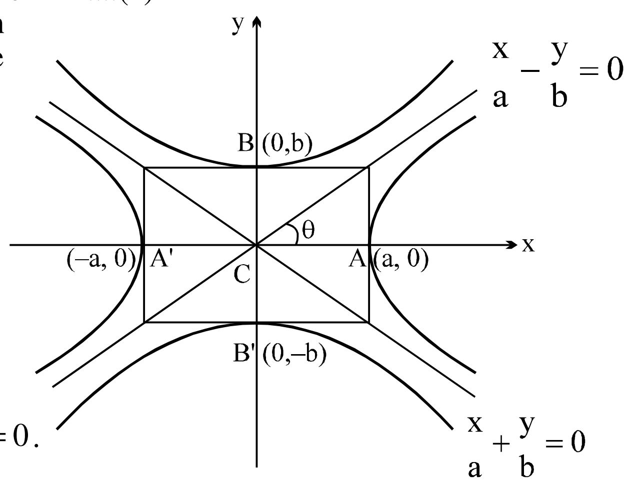
coeff of $x^2 = 0$ & coeff of $x = 0$.

$\Rightarrow \quad b^2 - a^2m^2 = 0$ or $m = \pm \frac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0$.

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.



combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

PARTICULAR CASE :

When b = a the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are, $y = \pm\ x$ which are at right angles.

Note :

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:
Let f (x, y) = 0 represents a hyperbola.
Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$
gives the centre of the hyperbola.

12. HIGHLIGHTS ON ASYMPTOTES:

H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

H–3 The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then $e = \sec\theta$.

13. RECTANGULAR HYPERBOLA :

Rectangular hyperbola referred to its asymptotes as axis of coordinates.

(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the points P (t_1) & Q(t_2) is $x + t_1t_2y = c(t_1 + t_2)$ with slope $m = -\frac{1}{t_1t_2}$.

(c) Equation of the tangent at P (x_1, y_1) is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P (t) is $\frac{x}{t} + ty = 2c$.

(d) Equation of normal : $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Suggested problems from Loney: **Exercise-36** (Q.1 to 6, 16, 22), **Exercise-37** (Q.1, 3, 5, 7, 12)

EXERCISE–7

Q.1 Find the equation to the hyperbola whose directrix is $2x + y = 1$, focus (1, 1) & eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

Q.2 The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

Q.3 For the hyperbola $\frac{x^2}{100} - \frac{y^2}{25} = 1$, prove that

(i) eccentricity $= \sqrt{5}/2$ (ii) SA. S′A = 25, where S & S′ are the foci & A is the vertex.

Q.4 Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes & the asymptotes of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

Q.5 The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ drawn at an extremity of its latus rectum is parallel to an asymptote. Show that the eccentricity is equal to the square root of $(1+\sqrt{5})/2$.

Q.6 If a rectangular hyperbola have the equation, $xy = c^2$, prove that the locus of the middle points of the chords of constant length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$.

Q.7 A triangle is inscribed in the rectangular hyperbola $xy = c^2$. Prove that the perpendiculars to the sides at the points where they meet the asymptotes are concurrent. If the point of concurrence is (x_1, y_1) for one asymptote and (x_2, y_2) for the other, then prove that $x_2y_1 = c^2$.

Q.8 The tangents & normal at a point on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cut the y − axis at A & B. Prove that the circle on AB as diameter passes through the foci of the hyperbola.

Q.9 Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Q.10 Ascertain the co-ordinates of the two points Q & R, where the tangent to the hyperbola $\frac{x^2}{45} - \frac{y^2}{20} = 1$ at the point P(9, 4) intersects the two asymptotes. Finally prove that P is the middle point of QR. Also compute the area of the triangle CQR where C is the centre of the hyperbola.

Q.11 If θ_1 & θ_2 are the parameters of the extremities of a chord through (ae, 0) of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then show that $\tan\frac{\theta_1}{2}\cdot\tan\frac{\theta_2}{2} + \frac{e-1}{e+1} = 0$.

Q.12 If C is the centre of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, S, S′ its foci and P a point on it.
Prove that SP. S′P = $CP^2 - a^2 + b^2$.

Q.13 Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

Q.14 If the tangent at the point (h, k) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the auxiliary circle in points whose ordinates are y_1 and y_2 then prove that $\frac{1}{y_1} + \frac{1}{y_2} = \frac{2}{k}$.

Q.15 Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x −axis. If $\tan\theta.\ \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

Q.16 If two points P & Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then prove that $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

Q.17 The perpendicular from the centre upon the normal on any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets at R. Find the locus of R.

Q.18 If the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P meets the transverse axis in G & the conjugate axis in g & CF be perpendicular to the normal from the centre C, then prove that $|PF.\ PG| = b^2$ & PF. Pg = a^2 where a & b are the semi transverse & semi-conjugate axes of the hyperbola.

Q.19 If the normal at a point P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the x − axis at G, show that SG = e. SP, S being the focus of the hyperbola.

Q.20 An ellipse has eccentricity 1/2 and one focus at the point P (1/2, 1). Its one directrix is the common tangent, nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. Find the equation of the ellipse in the standard form.

Q.21 Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.

Q.22 Prove that infinite number of triangles can be inscribed in the rectangular hyperbola, $xy = c^2$ whose sides touch the parabola, $y^2 = 4ax$.

Q.23 A point P divides the focal length of the hyperbola $9x^2 - 16y^2 = 144$ in the ratio S′P : PS = 2 : 3 where S & S′ are the foci of the hyperbola. Through P a straight line is drawn at an angle of 135° to the axis OX. Find the points of intersection of this line with the asymptotes of the hyperbola.

Q.24 Find the length of the diameter of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ perpendicular to the asymptote of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ passing through the first & third quadrants.

Q.25 The tangent at P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the asymptote in Q. Show that the locus of the mid point of PQ is a similiar hyperbola.

EXERCISE–8

Q.1 The chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose equation is $x \cos\alpha + y \sin\alpha = p$ subtends a right angle at the centre. Prove that it always touches a circle.

Q.2 If a chord joining the points P $(a\sec\theta, a\tan\theta)$ & Q $(a\sec\phi, a\tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that $\tan\phi = \tan\theta\,(4\sec^2\theta - 1)$.

Q.3 Prove that the locus of the middle point of the chord of contact of tangents from any point of the circle $x^2 + y^2 = r^2$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by the equation $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{(x^2 + y^2)}{r^2}$.

Q.4 A transversal cuts the same branch of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in P, P' and the asymptotes in Q, Q'. Prove that (i) PQ = P'Q' & (ii) PQ' = P'Q

Q.5 Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$. Also find the equation to the conjugate hyperbola & the equation of the principal axes of the curve.

Q.6 An ellipse and a hyperbola have their principal axes along the coordinate axes and have a common foci separated by a distance $2\sqrt{13}$, the difference of their focal semi axes is equal to 4. If the ratio of their eccentricities is 3/7. Find the equation of these curves.

Q.7 The asymptotes of a hyperbola are parallel to $2x + 3y = 0$ & $3x + 2y = 0$. Its centre is (1, 2) & it passes through (5, 3). Find the equation of the hyperbola.

Q.8 Tangents are drawn from any point on the rectangular hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that these tangents are equally inclined to the asymptotes of the hyperbola.

Q.9 The graphs of $x^2 + y^2 + 6x - 24y + 72 = 0$ & $x^2 - y^2 + 6x + 16y - 46 = 0$ intersect at four points. Compute the sum of the distances of these four points from the point (−3, 2).

Q.10 Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

Q.11 A series of hyperbolas is drawn having a common transverse axis of length 2a. Prove that the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymtote, is the curve $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$.

Q.12 A parallelogram is constructed with its sides parallel to the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and one of its diagonals is a chord of the hyperbola ; show that the other diagonal passes through the centre.

Q.13 The sides of a triangle ABC, inscribed in a hyperbola $xy = c^2$, makes angles α, β, γ with an asymptote. Prove that the nomals at A, B, C will meet in a point if $\cot 2\alpha + \cot 2\beta + \cot 2\gamma = 0$

Q.14 A line through the origin meets the circle $x^2 + y^2 = a^2$ at P & the hyperbola $x^2 - y^2 = a^2$ at Q. Prove that the locus of the point of intersection of the tangent at P to the circle and the tangent at Q to the hyperbola is curve $a^4(x^2 - a^2) + 4x^2y^4 = 0$.

Q.15 A straight line is drawn parallel to the conjugate axis of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to meet it and the conjugate hyperbola in the points P & Q. Show that the tangents at P & Q meet on the curve $\frac{y^4}{b^4}\left(\frac{y^2}{b^2} - \frac{x^2}{a^2}\right) = \frac{4x^2}{a^2}$ and that the normals meet on the axis of x.

Q.16 A tangent to the parabola $x^2 = 4ay$ meets the hyperbola $xy = k^2$ in two points P & Q. Prove that the middle point of PQ lies on a parabola.

Q.17 Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.

Q.18 Let 'p' be the perpendicular distance from the centre C of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the tangent drawn at a point R on the hyperbola. If S & S′ are the two foci of the hyperbola, then show that $(RS + RS')^2 = 4a^2\left(1 + \frac{b^2}{p^2}\right)$.

Q.19 P & Q are two variable points on a rectangular hyperbola $xy = c^2$ such that the tangent at Q passes through the foot of the ordinate of P. Show that the locus of the point of intersection of tangent at P & Q is a hyperbola with the same asymptotes as the given hyperbola.

Q.20 Chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle drawn on the line joining the foci as diameter. Find the locus of the point of intersection of tangents at the extremities of the chords.

Q.21 From any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to another hyperbola which has the same asymptotes. Show that the chord of contact cuts off a constant area from the asymptotes.

Q.22 The chord QQ′ of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is parallel to the tangent at P. PN, QM & Q′M′ are perpendiculars to an asymptote. Show that $QM \cdot Q'M' = PN^2$.

Q.23 If four points be taken on a rectangular hyperbola $xy = c^2$ such that the chord joining any two is perpendicular to the chord joining the other two and α, β, γ, δ be the inclinations to either asymptotes of the straight lines joining these points to the centre. Then prove that ; $\tan\alpha \cdot \tan\beta \cdot \tan\gamma \cdot \tan\delta = 1$.

Q.24 The normals at three points P, Q, R on a rectangular hyperbola $xy = c^2$ intersect at a point on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR.

Q.25 Through any point P of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a line QPR is drawn with a fixed gradient m, meeting the asymptotes in Q & R. Show that the product, $(QP) \cdot (PR) = \frac{a^2b^2(1+m^2)}{b^2 - a^2m^2}$.

EXERCISE–9

Q.1 Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$. [REE '97, 6]

Q.2 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$
(C) $x_1 x_2 x_3 x_4 = c^4$ (D) $y_1 y_2 y_3 y_4 = c^4$ [JEE '98, 2]

Q.3(a) The curve described parametrically by, $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents:
(A) a parabola (B) an ellipse (C) a hyperbola (D) a pair of straight lines

(b) Let P $(a \sec\theta, b\tan\theta)$ and Q $(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P & Q, then k is equal to:
(A) $\frac{a^2+b^2}{a}$ (B) $-\left(\frac{a^2+b^2}{a}\right)$ (C) $\frac{a^2+b^2}{b}$ (D) $-\left(\frac{a^2+b^2}{b}\right)$

(c) If x = 9 is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents, is :
(A) $9x^2 - 8y^2 + 18x - 9 = 0$ (B) $9x^2 - 8y^2 - 18x + 9 = 0$
(C) $9x^2 - 8y^2 - 18x - 9 = 0$ (D) $9x^2 - 8y^2 + 18x + 9 = 0$
[JEE '99, 2 + 2 + 2 (out of 200)]

Q.4 The equation of the common tangent to the curve $y^2 = 8x$ and $xy = -1$ is
(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$
[JEE 2002 Screening]

Q.5 Given the family of hyperbols $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ for $\alpha \in (0, \pi/2)$ which of the following does not change with varying α?
(A) abscissa of foci (B) eccentricity
(C) equations of directrices (D) abscissa of vertices [JEE 2003 (Scr.)]

Q.6 The line $2x + \sqrt{6}\,y = 2$ is a tangent to the curve $x^2 - 2y^2 = 4$. The point of contact is
(A) $(4, -\sqrt{6})$ (B) $(7, -2\sqrt{6})$ (C) (2, 3) (D) $(\sqrt{6}, 1)$ [JEE 2004 (Scr.)]

Q.7 Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of midpoint of the chord of contact. [JEE 2005 (Mains), 4]

Q.8(a) If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axis coincides with the major and minor axis of the ellipse, and product of their eccentricities is 1, then
(A) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (B) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(C) focus of hyperbola (5, 0) (D) focus of hyperbola is $\left(5\sqrt{3}, 0\right)$ [JEE 2006, 5]

Comprehension: (3 questions)
Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A

(a) If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is equal to
(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

(b) A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is
(A) ellipse (B) hyperbola (C) parabola (D) parts of straight line

(c) A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is
(A) 1/2 sq. units (B) 2/3 sq. units (C) 1 sq. unit (D) 2 sq. units
[JEE 2006, 5 marks each]

ANSWER KEY

PARABOLA

EXERCISE–1

Q.2 (a, 0) ; a **Q.3** $2x - y + 2 = 0$, (1, 4) ; $x + 2y + 16 = 0$, (16, −16)

Q.5 $3x - 2y + 4 = 0$; $x - y + 3 = 0$ **Q.6** (4 , 0) ; $y^2 = 2a(x - 4a)$

Q.8 $y = -4x + 72$, $y = 3x - 33$ **Q.9** $7y \pm 2(x + 6a) = 0$

Q.15 $x^2 + y^2 + 18x - 28y + 27 = 0$ **Q.18** $x - y = 1$; $8\sqrt{2}$ sq. units

Q.19 $\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$, vertex $\left(\frac{2}{9}, \frac{8}{9}\right)$ **Q.20** $15a^2/4$ **Q.21** (2a, 0) **Q.23** $a^2 > 8b^2$

EXERCISE–2

Q.3 $[a(t_o^2 + 4), -2at_o]$ **Q.5** $(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$

Q.10 (a) $\left(-\frac{1}{2}, \frac{1}{4}\right)$; (b) $y = -(x^2 + x)$ **Q.12** $((x_1 - 2a), 2y_1)$ **Q.21** $y^2 = 8ax$

Q.16 Q(4, −8) **Q.18** $(x^2 + y^2 - 4ax)^2 = 16a(x^3 + xy^2 + ay^2)$

EXERCISE–3

Q.1 $x^2 - 2y + 12 = 0$ **Q.3** $x = 3\left[7\left(\frac{y}{18}\right)^{2/3} + 2\right]$ **Q.4** $x - 2y + 1 = 0$; $y = mx + \frac{1}{4m}$ where $m = \frac{-5 \pm \sqrt{30}}{10}$

Q.5 (a) C ; **(b)** B **Q.6** $(x + 3)y^2 + 32 = 0$ **Q.7** **(a)** C ; **(b)** D **Q.8** C

Q.9 D **Q.10** **(a)** C ; **(b)** $\alpha = 2$ **Q.11** B

Q.12 $xy^2 + y^2 - 2xy + x - 2y + 5 = 0$ **Q.13** (a) D, (b) A, B, (c) (i) A, (ii) B, (iii) D, (iv) C

ELLIPSE

EXERCISE–4

Q.1 $20x^2 + 45y^2 - 40x - 180y - 700 = 0$ **Q.4** $3x^2 + 5y^2 = 32$

Q.8 $x + y - 5 = 0$, $x + y + 5 = 0$ **Q.9** $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$; $4x \pm \sqrt{33}\,y - 32 = 0$

Q.10 24 sq.units **Q.11** $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **Q.14** $55\sqrt{2}$ sq. units **Q.19** $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$

EXERCISE–5

Q.1 $(a^2 - b^2)^2 x^2y^2 = a^2(a^2 + b^2)^2 y^2 + 4b^6x^2$ **Q.5** $bx + a\sqrt{3}\,y = 2ab$

Q.13 $\sqrt{r^2 - b^2}$ **Q.15** $12x + 5y = 48$; $12x - 5y = 48$

EXERCISE–6

Q.1 $\phi = \pi - \tan^{-1}2$, $t = -\frac{1}{\sqrt{5}}$; $\phi = \pi + \tan^{-1}2$, $t = \frac{1}{\sqrt{5}}$; $\phi = \pm\frac{\pi}{2}$, $t = 0$

Q.3 **(i)** C ; **(ii)** C **Q.4** **(a)** A ; **(b)** B, D ; **(c)** $25\,y^2 + 4\,x^2 = 4\,x^2 y^2$ **Q.5** $(x-1)^2 + y^2 = \frac{11}{3}$

Q.7 Locus is an ellipse with foci as the centres of the circles C_1 a nd C_2.

Q.8 $a^2p^2 + b^2q^2 = r^2\sec^2\frac{\pi}{8} = (4 - 2\sqrt{2})r^2$ **Q.10** **(a)** C ; **(b)** A **Q.11** C **Q.12** (a) A, (b) $AB = \frac{14}{\sqrt{3}}$

HYPERBOLA

EXERCISE–7

Q.1 $7\,x^2 + 12xy - 2\,y^2 - 2x + 4y - 7 = 0$; $\sqrt{\frac{48}{5}}$ **Q.2** $a^2 = 25/2$; $b^2 = 16$

Q.4 $(-1, 2)$; $(4, 2)$ & $(-6, 2)$; $5x - 4 = 0$ & $5x + 14 = 0$; $\frac{32}{3}$; 6 ; 8 ; $y - 2 = 0$;
$x + 1 = 0$; $4x - 3y + 10 = 0$; $4x + 3y - 2 = 0$.

Q.9 $x + y \pm 3\sqrt{3} = 0$ **Q.10** (15, 10) and (3, −2) and 30 sq. units

Q.13 $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$ **Q.17** $(x^2 + y^2)^2\,(a^2y^2 - b^2x^2) = x^2y^2\,(a^2 + b^2)^2$

Q.20 $\frac{\left(x-\frac{1}{3}\right)^2}{1/9} + \frac{(y-1)^2}{1/12} = 1$ **Q.23** $(-4, 3)$ & $\left(-\frac{4}{7}, -\frac{3}{7}\right)$ **Q.24** $\frac{150}{\sqrt{481}}$ **Q.25** $4\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 3$

EXERCISE–8

Q.5 $x - 2y + 1 = 0$; $2x + y + 1 = 0$; $2x^2 - 3xy - 2y^2 + 3x - y - 6 = 0$; $3x - y + 2 = 0$; $x + 3y = 0$

Q.6 $\frac{x^2}{49} + \frac{y^2}{36} = 1$; $\frac{x^2}{9} - \frac{y^2}{4} = 1$ **Q.7** $6x^2 + 13xy + 6y^2 - 38x - 37y - 98 = 0$

Q.9 40 **Q.10** $y = \frac{5}{12}x + \frac{3}{4}$; $x - 3 = 0$; 8 sq. unit

Q.19 $xy = \frac{8}{9}c^2$ **Q.20** $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}$ **Q.21** ab

EXERCISE–9

Q.1 $(x^2 + y^2)^2 = 16\,x^2 - 9\,y^2$ **Q.2** A, B, C, D **Q.3** **(a)** A ; **(b)** D ; (c) B **Q.4** D

Q.5 A **Q.6** A **Q.7** $\frac{x^2}{9} - \frac{y^2}{4} = \left(\frac{x^2+y^2}{9}\right)^2$ **Q.8** (a) A, (b) C, (c) C